Abstract—This paper introduces an Explicit Reference Governor to supervise closed-loop linear time-delay systems. The proposed scheme enforces state and input constraints by modifying the reference of the supervised system so that the state vector always belongs to admissible sub-level sets of a suitably defined Lyapunov-Krasovskii functional. To accomplish this, the paper extends the existing definition of “dynamic safety margin” to a time-delay setting and illustrates how to employ classic Lyapunov-Krasovskii functionals even though the reference is time-varying. Constraint enforcement for arbitrary reference signals and asymptotic convergence to any strictly steady-state admissible setpoint is rigorously proven. Experimental results are reported to demonstrate the simplicity, practicality, and robustness of the proposed method.

I. INTRODUCTION

In many real-world applications, the presence of propagation and transport phenomena affecting the process, sensors, actuators, or communication channels can lead to a non-instantaneous response of the dynamic system, also known as a delay. The trajectory of a time-delay system is determined by the past history of the states and inputs rather than just the current state. As a result, these systems are infinite-dimensional and their treatment presents distinct challenges that require different tools with respect to conventional systems.

The stability and feedback stabilization of unconstrained time-delay systems is the subject of a vast literature, see e.g. [1]–[4]. In the presence of constraints, several a posteriori analysis tools have been proposed which allow to characterize and possibly tune the admissible set of initial conditions for a stabilized linear time-delay system. Examples can be found in [5]–[8], which characterize the basin of attraction in the presence of input saturation, and [9], [10], which construct polyhedral invariant sets for a closed-loop time-delay system subject to state constraints. However, the performance of these approaches is somewhat limited by the fact that the control action does not actively account for constraints.

A possible alternative for the control of constrained time-delay systems is the use Model Predictive Control (MPC), which generates the control action based on the solution of a constrained optimal control problem which is repeatedly solved online [11]–[14]. Although these schemes typically achieve high performances in terms of output response, a possible drawback of MPC is that the required computational cost may be too prohibitive for certain applications. In the presence of only input constraints, anti-windup strategies such as [15], [16] have been proposed for time-delay system and can be used to drastically reduce computations.

A solution that is somewhat in-between MPC and anti-windup is the Reference Governor (RG). RG schemes for linear time-delay systems subject to state and input constraints can be found in [17]–[21]. Although the RG approach tends to be less computationally intensive than MPC due to its particular formulation, see e.g. [22], the computational cost may still be problematic for some real-time applications. For systems without delay, the recently introduced Explicit Reference Governor (ERG) overcomes this issue by providing a closed-form implementation strategy that does not require the solution of an online optimization problem [23], [24]. The objective of this paper is to extend the ERG framework to the case of linear time-delay systems, thus enabling the applicability of the scheme in practical applications where the effects of delays are not negligible.

The paper is structured as follows: Section II introduces the overall control problem and reformulates it in terms of two sub-problems, i.e. stabilization and constraint enforcement. Section III provides a brief summary of existing results pertaining to the stabilization of linear time-delay systems. Section IV then illustrates how these results can be used in an ERG setting to ensure constraint enforcement. In particular, it is shown how the classical ERG theory, which typically relies on Lyapunov functions, can be modified to employ Lyapunov-Krasovskii functionals which depend on the past history of the system. In Section V, the proposed scheme is validated experimentally on an industrially relevant example. Finally, Section VI concludes the paper with some closing remarks.

II. PROBLEM STATEMENT

In this paper we consider Linear Time-Delay (LTD) systems with input delay of the form,
\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t - \tau), \\
y(t) &= C x(t) + D u(t - \tau),
\end{align*}
\]  
(1)
where \(x(t) \in \mathbb{R}^n\) is the state, \(u(t) \in \mathbb{R}^m\) is the control input, \(y(t) \in \mathbb{R}^p\) is the output, \(\tau > 0\) is a known constant delay, and \((A, B, C, D)\) are suitably dimensioned matrices such that the system is stabilizable. System (1) is subject to linear inequality constraints on both the state and input vectors,
\[
h_{x,i}^T x(t) + h_{u,i}^T u(t) + g_i \geq 0, \quad i = 1, \ldots, n_c,
\]  
(2)
where \(h_{x,i} \in \mathbb{R}^n, h_{u,i} \in \mathbb{R}^m,\) and \(g_i \in \mathbb{R}\) are assumed to generate a non-empty full-dimensional polyhedral set of state and input constraints. Note that (2) can be used to represent pure state constraints, whenever \(h_{u,i} = 0\), pure input saturation constraints, whenever \(h_{x,i} = 0\), as well as mixed state and input constraints. Given an output reference signal \(r(t) \in \mathbb{R}^p\), and given suitable initial conditions, the goal of this paper is to design a closed-form control strategy which satisfies the following objectives:

1. For any piecewise continuous signal \(r(t)\), not known in advance, constraints (2) are satisfied at all times;
2. If there exists \(t_1\) such that \(r(t) = r\) for all \(t \geq t_1\), and if the reference \(r\) is consistent with the constraints (2), then \(\lim_{t \to \infty} y(t) = r\).
The objective of the primary control unit is to pre-stabilize the system. The objective of the primary control unit can be stated as:

**Problem 1:** Consider a constant reference \( v \in \mathbb{R}^p \), and let the steady-state configuration \( \bar{x}_v := G_x v, \bar{u}_v := G_u v \) satisfy
\[
A\bar{x}_v + B\bar{u}_v = 0, \quad C\bar{x}_v + D\bar{u}_v = v.
\]

The objective of the primary control unit is to design a control law such that the equilibrium point \( \bar{x}_v, \bar{u}_v \) is Globally Asymptotically Stable (GAS).

The objective of the auxiliary control loop is as follows.

**Problem 2:** Given the pre-stabilized system obtained as the solution to Problem 1, and given suitable initial conditions \(^1\) \( u_v(0), x(0), v(0) \), generate an auxiliary reference \( v(t) \in \mathbb{R}^p \) such that:
1. For any piecewise continuous reference signal \( r(t) \in \mathbb{R}^p \), the trajectories of \( x(t) \), \( u(t) \) satisfy constraints (2).
2. For any constant reference signal \( r(t) = r \in \mathbb{R}^p \) such that \( \bar{x}_v, \bar{u}_v \) strictly satisfy constraints (2), \( v(t) \) asymptotically tends to \( r \).

The following section addresses the design of the primary control unit using well-known results from the existing literature, which are reported for the reader’s convenience. Particular emphasis is given to the construction of suitable Lyapunov-Krasovskii functionals, which will be used as a starting point for the auxiliary control unit addressed in Section IV.

### III. Primary Control

The objective of the primary control unit is to pre-stabilize the system dynamics in the absence of constraints. Given a constant reference \( v \) and a linear control law in the form
\[
u(t) = G_u v + K(x(t) - \bar{x}_v),
\]
the dynamics of the closed-loop system satisfy
\[
\dot{e}(t) = A e(t) + B K e(t - \tau),
\]
where \( e(t) = x(t) - \bar{x}_v \). Problem 1 can therefore be reduced to finding a control gain \( K \) such that the origin of (6) is GAS. This can be done using the techniques described in, e.g., [3], [4]. The stability analysis of LTD systems can also be found in existing literature. For the reader’s convenience, however, the following subsections briefly summarize some of the results detailed in [25] and references therein. These results will be used as a starting point for the design of the auxiliary control unit in Section IV.

#### A. Stability Conditions without Delay

In the absence of time-delays, i.e., \( \tau = 0 \), it is well-known that \( \bar{x}_v \) is GAS if and only if there exists a quadratic function
\[
V(e) = e^T P e,
\]
such that \( P > 0 \) satisfies the Linear Matrix Inequality (LMI)
\[
(A + BK)^T P + P (A + BK) < 0.
\]
As detailed in the following subsections, a similar approach can be used in the presence of time-delays.

#### B. Delay-Independent Conditions

The main difficulty in LTD systems is that, unlike the case \( \tau = 0 \), the literature does not provide a necessary and sufficient condition for proving GAS for the case \( \tau > 0 \). Instead, stability is typically proven on a case-by-case basis by trying to verify the validity of various candidate Lyapunov-Krasovskii functionals. A possible example is the candidate functional
\[
V(e_r(t)) = e(t)^T P e(t) + \int_{t-\tau}^{t} e(s)^T Q e(s) ds,
\]
where \( P > 0, Q > 0 \) are required to satisfy the LMI
\[
\begin{bmatrix}
A^T P + PA + Q & PBK \\
PBK^T & -Q
\end{bmatrix} < 0.
\]

The idea behind equation (9) is that the fluctuations of the classical quadratic term (7) may be compensated using an integral term that takes into account the “memory” of the system. If the LMI (10) admits a solution, it follows that GAS can be proven using the Lyapunov-Krasovskii functional (9). Otherwise, nothing can be said about the system stability.

One of the appealing properties of the candidate Lyapunov-Krasovskii functional (9) is that the underlying LMI does not depend on \( \tau \), meaning that it proves GAS even in the case of time-varying delays. However, it is worth noting that the LMI (10) admits a solution only if there exists \( S > 0 \) such that \( A^T S + SA < 0 \). As a result, this approach intrinsically requires the open-loop system (1) to already be GAS. The following subsection provides a possible way to address open loop unstable systems.

#### C. Delay-Dependent Conditions

In many applications, it is common to observe that a closed-loop system may become unstable given an excessive time-delay. As a result, for many LTD systems it is reasonable to assume that the stability conditions should also depend on the value \( \tau \). In this setting, a well-known (albeit simple) approach for proving GAS is the use of the candidate Lyapunov-Krasovskii functional
\[
V(e_r(t)) = e(t)^T P e(t) + \int_{t-\tau}^{t} \left( \theta - t + \tau \right) e(\theta)^T R e(\theta) d\theta,
\]
where \( P > 0, R > 0 \) are required to satisfy the LMI
\[
\begin{bmatrix}
\Phi & -\Psi_1^T + (A + BK)^T \Psi_3 \\
* & -\Psi_3^T + \tau R
\end{bmatrix} < 0,
\]
where \( \Phi = (A + BK)^T \Psi_2 + \Psi_2^T (A + BK) \), and \( \Psi_2, \Psi_3 \) are slack variables. Once again, it is worth noting that the existence of a Lyapunov-Krasovskii functional in the form (11) is only a sufficient condition for GAS. Several extensions can be found in the literature.
see e.g. [26]. Delay-dependent Lyapunov-Krasovskii functionals have also been proposed for certain special classes of time-varying delay systems. Notable examples include sampled-data systems, where the time-delay is modeled as a sawtooth function [27]. Note that the constraint enforcement strategy presented in the next section can be easily applied to [26], [27] and many other LTD stability results using arguments analogous to the ones developed for (11).

IV. AUXILIARY CONTROL

The objective of this section is to propose a novel Explicit Reference Governor to augment a pre-stabilized LTD system so that constraint satisfaction is ensured. Inspired by [24], which addresses the problem in the absence of time-delays, the proposed ERG generates the auxiliary reference \( v(t) \) as follows

\[
\dot{v}(t) = \Delta(x_r(t), v_r(t))\rho(v(t), r(t)),
\]

where the function \( \rho : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^p \) is an “Attraction Field” as per [24, Definition 2]. The following definition extends the concept of “Dynamic Safety Margin” introduced in [24] to account for the memory of LTD systems

**Definition 1: Dynamic Safety Margin for LTD Systems.** Let the pre-stabilized system (6) be subject to constraints (2), let \( \dot{v}(\theta | x_r, v_r) \) be

\[
\begin{cases}
\dot{v}(\theta) = v(t), & \forall \theta \geq 0, \\
\dot{v}(\theta) = v(t + \theta), & \forall \theta \in [-\tau, 0),
\end{cases}
\]

and let \( \dot{x}(\theta | x_r(t), v_r(t)) \) denote the solution to

\[
\begin{cases}
\dot{x}(\theta) = A\dot{x}(\theta) + Bu(\theta - \tau), & \forall \theta \geq 0, \\
\dot{x}(\theta) = x(t + \theta), & \forall \theta \in [-\tau, 0),
\end{cases}
\]

with

\[
\dot{u}(\theta) = G_u\dot{v}(\theta) + K(\dot{x}(\theta) - G_x\dot{v}(\theta)), & \forall \theta \geq -\tau.
\]

Given an auxiliary reference \( v \) strictly satisfying constraints (2), i.e.

\[
h_{x,i}^T\dot{x} + h_{u,i}^T\dot{u} + g_i > 0, \quad i = 1, \ldots, n_c,
\]

a continuous functional \( \Delta : \mathbb{R}^n_{[v, r]} \times \mathbb{R}^n_{[x, \theta]} \rightarrow \mathbb{R} \) is a Dynamic Safety Margin if the following properties hold

\[
\begin{align*}
\Delta(x_r(t), v_r(t)) &\geq 0 \quad \Rightarrow \quad \Delta(\dot{x}(\theta), \dot{v}(\theta)) \geq 0, \quad (18a) \\
\Delta(x_r(t), v_r(t)) &\geq 0 \quad \Rightarrow \quad h_{x,i}^T\dot{x} + h_{u,i}^T\dot{u} + g_i > 0, \quad (18b) \\
\Delta(x_r(t), v_r(t)) &\geq 0 \quad \Rightarrow \quad h_{x,i}^T\dot{x} + h_{u,i}^T\dot{u} + g_i > 0, \quad (18c) \\
h_{x,i}^T\dot{x} + h_{u,i}^T\dot{u} + g_i \geq \delta &\Rightarrow \quad \Delta(\dot{x}(\theta), \dot{v}(\theta)) \geq \varepsilon, \quad (18d)
\end{align*}
\]

for all \( \theta \geq 0 \), for all \( i \in \{1 : n_c\} \), and suitable \( \varepsilon > 0, \delta > 0 \). □

The idea behind the ERG expression (13) is that \( \rho(v, r) \) identifies the direction in which \( v \) should evolve, whereas \( \Delta(x_r, v_r) \) estimates the distance between the constraints and the future system trajectories if the auxiliary reference \( v \) were to be kept constant from the time \( t \) onward. This formulation ensures that, whenever \( \Delta(x_r, v_r) = 0 \), constraint satisfaction can be guaranteed by maintaining \( \dot{v} = 0 \). Likewise, whenever \( \Delta(x_r, v_r) > 0 \), it is possible to assign \( \|\dot{v}\| \neq 0 \) without running the risk of violating constraints. The following subsections illustrate how to construct a suitable dynamic safety margin and attraction field for LTD systems subject to linear constraints.

A. Dynamic Safety Margin

A possible way to satisfy requirements (18) is to compute the minimal distance between the constraints and the predicted trajectory at all future time instants, which leads to

\[
\Delta(x_r, v_r) = \min_{\theta \geq 0} \left\{ h_{x,i}^T\dot{x}(\theta) + h_{u,i}^T\dot{u}(\theta) + g_i \right\}. 
\]

The main issue with equation (19) is that the trajectories \( \dot{x}(\theta), \dot{u}(\theta) \) would have to be computed over the infinite horizon \( \theta \geq 0 \). As detailed in the following proposition, this limitation can be overcome by computing the trajectory over a finite window of duration \( T \geq \tau \) and bounding the remaining trajectories using the level sets of Lyapunov-Krasovskii functionals.

**Proposition 1:** Consider the trajectories (14)-(16), and let \( T \geq \tau \) be a prediction horizon. Given

\[
\Delta_T(x_r, v_r) = \max_{\theta \in [0, T]} \left\{ h_{x,i}^T\dot{x}(\theta) + h_{u,i}^T\dot{u}(\theta) + g_i \right\}, 
\]

and given

\[
\Delta_{\infty}(x_r, v_r) = \Gamma(v) - V(\dot{x}(\theta)),
\]

where \( V(\dot{x}(\theta)) \) is a Lyapunov-Krasovskii functional evaluated at \( \dot{x}(\theta) = \dot{x}(\tau) - v \), and \( \Gamma(v) \) is a threshold value satisfying

\[
V(\dot{x}(\theta)) \leq \Gamma(v) \Rightarrow h_{x,i}^T\dot{x} + h_{u,i}^T\dot{u} + g_i \geq 0,
\]

for \( i \in \{1 : n_c\} \), then

\[
\Delta(x_r, v_r) = \min_{\epsilon \in \{1 : n_c\}} \left\{ h_{x,i}^T\dot{x}(\theta) + h_{u,i}^T\dot{u}(\theta) + g_i \right\}, 
\]

is a dynamic safety margin for any \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \). □

**Proof:** Since equation (23) takes into account the worst-case scenario, the satisfaction of (18) can be addressed separately over the time windows \( \theta \in [0, T] \) and \( \theta \in (T, \infty) \).

For what concerns the time window \( \theta \in [0, T] \), it is sufficient to note that (20) is equal to the minimum distance between the trajectory of (14)-(16) and the boundary of constraints (2). As a result, it satisfies (18) by construction.

As for the time window \( \theta \in (T, \infty) \), it follows from (15) that \( \dot{v}(\theta) = v \) for all \( \theta \in [T - \tau, \infty) \). Due to the requirement \( T \geq \tau \), this implies that (14)-(16) is consistent with the dynamics of an LTD system subject to a constant reference for all \( \theta \geq T \). As a result, the Lyapunov-Krasovskii functionals detailed in Section III are applicable and therefore satisfy \( V(\dot{x}(\theta)) \leq V(\dot{x}(\tau)) \), all \( \theta > T \). The satisfaction of (18) thus follows from (21)-(22). ■

**Remark 1:** The main challenge addressed by Proposition 1 is that, unlike classical ERG schemes [23], [24], the Lyapunov-Krasovskii functionals (9), (11) are not applicable at time \( t \) because the auxiliary reference \( v(t) \) is not constant for \( t \in [t - \tau, t] \). This challenge was overcome by generating the virtual auxiliary reference (14) and using trajectory predictions to pad the memory of the system so that (9), (11) are valid for \( \theta = T \). □

**Remark 2:** Equation (14) does not imply that the auxiliary reference will (or should) satisfy \( v(t + \theta) = \tilde{v}(\theta) \), \( \forall \theta \in (0, \infty) \). Instead, the objective of Proposition 1 is to recursively ensure that constraint satisfaction can always be enforced by assigning \( \dot{v} = 0 \).

The final step for determining (23) is to compute a suitable threshold value \( \Gamma(v) \), which can be interpreted as the value of a level-set that is entirely contained in the constraints. To do so, it is sufficient to note that the proposed Lyapunov-Krasovskii functionals (9), (11) both satisfy the quadratic lower bound

\[
V(\dot{x}(\theta)) \geq \dot{v}(\theta)^TP\dot{v}(\theta).
\]

(24)
As a result, it follows from [28] that, for any given linear constraint (2), equation (22) can be satisfied for
\[
\Gamma_i(v) = \frac{(h_{x,i}^T \bar{x}_v + h_{u,i}^T \bar{u}_v + g_i)^2}{(h_{x,i} + h_{u,i} K)^P^{-1}(h_{x,i} + K^T h_{u,i})}.
\] (25)

As a consequence, the collection of all the constraints can be taken into account by choosing \(\Gamma(v) = \min_i \{\Gamma_i(v)\} \) for \(i \in \{1 : n_c\}\). Once again, it is worth noting that this approach is valid for any Lyapunov-Krasovskii functional that satisfies (24). Since this property holds for many LMI results, e.g., all the formulations provided in [25], the constraint handling strategy presented in this paper is easily generalizable to account for more advanced Lyapunov-Krasovskii functionals.

**Remark 3:** It is worth noting that the matrix \(P\) should be chosen so that the volume of the largest Lyapunov level-set compatible with the constraints is as aligned as possible to the constraint boundary while also satisfying (8). Although in the case of time-delay systems it is somewhat less clear how the choice of \(P\) influences the performance of the ERG, a practical approach for selecting the matrix \(P\) (as well as all the remaining auxiliary variables \(Q > 0, R > 0\)) is to solve the optimization problem (26), where the last line should be substituted with (10) or (12) depending on the LMI associated to the proposed Lyapunov-Krasovskii functional.

\[
\begin{align*}
\min & \; \log \det P \\
\text{s.t.} & \; P \geq (h_a + K^T h_u)(h_a^T + h_u^T K), \\
& \; P > 0, \\
& \; (A + BK)^T P + P(A + BK) < 0.
\end{align*}
\] (26)

This is justified by the idea that \(P\) should be chosen so that the volume of the largest Lyapunov level-set compatible with the constraints is as aligned as possible to the constraint boundary while also satisfying (8). Although in the case of time-delay systems it is somewhat less clear how the choice of \(P\) influences the performance of the ERG, a practical approach for selecting the matrix \(P\) (as well as all the remaining auxiliary variables \(Q > 0, R > 0\)) is to solve the optimization problem (26), where the last line should be substituted with (10) or (12) depending on the LMI associated to the proposed Lyapunov-Krasovskii functional.

### B. Attraction Field

This subsection addresses the construction of the attraction field \(\rho(v, r)\). Since this element of the ERG does not depend on the system delays, it is possible to use the standard attraction/repulsion approach [29]. Therefore, a possible solution is to use
\[
\rho(v, r) = \rho_0(v, r) + \sum_{i=1}^{n_c} \rho_i(v),
\] (27)
where
\[
\rho_0(v, r) = \frac{r - v}{\min \{\|v - r\|, \eta\}},
\] (28)
with \(\eta > 0\), is an attraction term that points towards \(r \in \mathbb{R}^p\), and
\[
\rho_i(v) = \max \left\{ \frac{\zeta - \delta}{\|h_{x,i} + h_{u,i} K\|}, 0 \right\}
\]
with \(\zeta > \delta > 0\), is a repulsion term associated to the \(i\)th constraint.

### C. Main Result

The following proposition formally states the properties of the proposed constrained control architecture.

**Proposition 2:** Let (1) be a time-delayed linear system subject to a constant delay \(\tau\), and let (2) be a nonempty set of linear state and input constraints. Given a primary control law (5), let \(V(e_v)\) be a Lyapunov-Krasovskii functional and let (23) be the associated dynamic safety margin. Then, given the attraction field (27) and an initial auxiliary reference \(v(0)\) such that \(\Delta(e_v(0), v(0)) \geq 0\), the Explicit Reference Governor (13) ensures the following properties:

1. For any piecewise continuous reference \(r(t) \in \mathbb{R}^p\), constraints (2) are satisfied for all time instants \(t \geq 0\);
2. For any constant reference \(r \in \mathbb{R}^p\) such that \(\bar{x}_r, \bar{u}_r\) strictly satisfy (2), the system output \(y(t)\) asymptotically tends to \(r\).

**Proof:** To prove Point 1, consider the following cases:

- Whenever \(\Delta(x_r, v_r) > 0\), it follows from Proposition 1 that if the current reference were to remain constant, the future system trajectories would strictly satisfy constraints (2). Therefore, it follows by continuity that \(\dot{v} > 0\) cannot cause a constraint violation.
- Whenever \(\Delta(x_r, v_r) = 0\), it follows from Proposition 1 that if the current reference remains constant, the future system trajectories will not violate constraints (2). Constraint enforcement is thus ensured by equation (13) due to which \(\Delta(x_r, v_r) = 0 \Rightarrow \dot{v} = 0\).

Due to equation (13) and the time-decreasing nature of Lyapunov-Krasovskii functionals, the case \(\Delta(x_r, v_r) < 0\) cannot occur as long as the system is correctly initialized.

Point 2 of the statement can be proven as in [24, Theorem 1], which relies on the same attraction field as the one used in this paper.

It is worth noting that the systematic implementation of the ERG framework to time-delayed systems is only limited by the availability of a suitable Lyapunov-Krasovskii functionals proving the stability of the primary control loop. The following section illustrates how the results presented in this paper can be implemented in practice and also shows that the overall performance of the controlled system will depend on the selected Lyapunov-Krasovskii functional.

### V. EXPERIMENTAL VALIDATION

To validate the results obtained in this paper, the proposed ERG strategy is implemented on an experimental setup composed of a Kobold type DF-MA flow rate sensor and of a Burkert type 8605 actuated valve (see Fig. 2). The identified dynamics for the open-loop system are
\[
\begin{align*}
\dot{x}(t) &= ax(t) + bu(t - \tau), \\
y(t) &= x(t)
\end{align*}
\]
where \(x\) is the water flow rate [l/h], \(u\) is the open percentage of the valve [%], \(\tau = 0.8\) [s] is the delay, \(a = -0.82\) [s\(^{-1}\)], and \(b = 0.7279\) [l/h\(^{-1}\)s\(^{-1}\)]. The system is required to go from the
Fig. 3. Experimental results of the closed-loop response for the case \( k = -1 \). The dashed lines represent the auxiliary references \( v(t) \), whereas the solid lines represent the state \( x(t) \). The constraint boundary is represented by the dotted red line.

starting condition \( x(0) = 0 \) [l/h] to the desired set-point \( r = 26 \) [l/h] without violating the constraint \( x \leq 26.6 \) [l/h]. The proposed primary control law is
\[
u(t) = \bar{u}_v + k(x(t) - v),\]
where \( \bar{u}_v \) was identified experimentally. The stability of the closed-loop system depends on the value of the control gain \( k \). In particular, the following cases hold:
- For any \( k \in [-1.12, 1.12] \), the LMI (10) admits a solution, thus implying that (9) is a suitable Lyapunov functional;
- For any \( k \in [-1.77, 0] \), the LMI (12) admits a solution, thus implying that (11) is a suitable Lyapunov functional;
- For any \( k \in (-3.54, -1.77) \), asymptotic stability may be proven using other Lyapunov functionals available in the LTD literature, see e.g. [25];
- For any \( k < -3.54 \) or \( k > 1.12 \), the system is unstable.

The behavior of the ERG was tested for two different values of the control gain for which the stability of the pre-stabilized system can be proven using the Lyapunov-Krasovskii functionals considered in this paper.

A. Mild Control Action

Given \( k = -1 \), the Lyapunov functionals (9), and (11) hold for \( P = 1, Q = 0.86 \), and \( R = 0.95 \). The experimental results are detailed in Figure 3, which illustrate the behavior of the auxiliary reference and of the system output for the following cases:
- **No Reference Governor (No RG):** The closed-loop system is directly subject to a step variation of the desired reference;
- **“Infinite” Horizon ERG (ERG-0):** The auxiliary reference is issued by an ERG, where (23) is computed with \( T = 7s \) and \( \kappa_1 = 50 \). Due to the sizable length of the prediction horizon, the terminal dynamic safety margin \( \Delta_\infty \) is omitted;
- **ERG, option 1 (ERG-1):** The auxiliary reference is issued by an ERG, where (23) is computed with \( T = 0.7s, \kappa_1 = 50 \), \( \kappa_2 = 20 \). The terminal dynamic safety margin \( \Delta_\infty \) is based on the Lyapunov-Krasovskii functional (9);
- **ERG, option 2 (ERG-2):** The auxiliary reference is issued by an ERG, where (23) is computed with \( T = 0.7s, \kappa_1 = 50, \kappa_2 = 20 \). The terminal dynamic safety margin \( \Delta_\infty \) is based on the Lyapunov-Krasovskii functional (11).

B. Aggressive Control Action

Given \( k = -1.68 \), it is not possible to prove asymptotic stability using delay-independent conditions. However, the Lyapunov functional (11) holds for \( P = 1 \) and \( R = 0.64 \). Figure 4 details the experimental results for the following cases:
- **No Reference Governor (No RG):** The closed-loop system is directly subject to a step variation of the desired reference;
- **“Infinite” Horizon ERG (ERG-0):** The auxiliary reference is issued by an ERG, where (23) is computed with \( T = 7s \) and \( \kappa_1 = 50 \). Due to the sizable length of the prediction horizon, the terminal dynamic safety margin \( \Delta_\infty \) is omitted;
- **ERG, option 2 (ERG-2):** The auxiliary reference is issued by an ERG, where (23) is computed with \( T = 0.7s, \kappa_1 = 50, \kappa_2 = 20 \). The terminal dynamic safety margin \( \Delta_\infty \) is based on the Lyapunov-Krasovskii functional (11).

C. Discussion

As can be observed in Figures 3 and 4, the dynamic response of the pre-stabilized system is characterized by a delay-induced overshoot that leads to a violation of the constraints. The proposed ERG strategies successfully overcome this issue by suitably modulating the dynamics of the auxiliary reference. This result is achieved despite the presence of numerous non-ideal conditions that are a common occurrence in practical applications. In particular, in addition to the typical presence of parameter uncertainties, sensor noise, and input disturbances, we have observed that the time-delay fluctuates in a range of \( \pm 6\% \) around the nominal value \( \tau = 0.8 \) [s]. This demonstrates that the proposed scheme is sufficiently robust to
handle mild violations of the assumptions under which it is derived, notably the fact that $\tau$ is a known and constant parameter.

In terms of comparisons between alternative ERG formulations, it is worth noting that the ERG-0, i.e. the “infinite” horizon approach, achieves the fastest response out of all the ERG strategies. This was to be expected since the use of trajectory predictions leads to a fairly aggressive control strategy. Nevertheless, the method has a few practical disadvantages that may limit its applicability. First of all, the trajectory predictions may be inaccurate depending on the quality of the model. Consequently, constraint violations are more likely to occur. Second, the computational cost of performing the predictions for higher-order systems may be too restrictive for real-time implementation. By contrast, the Lyapunov-Krasovskii-based schemes, i.e. ERG-1 and ERG-2, are more robust to model uncertainties, due to the inherent conservativeness of Lyapunov-based invariant sets, and are also less computationally expensive, due to the fact that they do not require trajectory predictions over a long horizon. This is counterbalanced by a decrease in the output performance. Additionally, it is worth noting that the loss of performance greatly depends on the quality of the invariant level-set that is used. This is most apparent in Figure 3, where we note that the performance of the ERG-2 is fairly similar to the ERG-0, whereas the output response of the ERG-1 is significantly slower. This suggests that the performance of the proposed scheme is affected by the choice of the underlying Lyapunov-Krasovskii functional. This observation is consistent with the results presented in [28] for linear time-invariant systems without delays, where it is shown that, in the presence of active constraints, the convergence rate depends on the size of the invariant set used to ensure constraint enforcement.

VI. CONCLUSIONS

This paper proposed an Explicit Reference Governor approach for the control of time-delay systems subject to state and input constraints. The method consists in pre-stabilizing the system using a primary control loop and then introducing an auxiliary control loop that ensures constraint satisfaction by suitably manipulating the dynamics of the applied reference. The proposed scheme can be implemented using Lyapunov-Krasovskii functionals from the existing literature. Experimental validation demonstrates the effectiveness and robustness of the proposed method on an industrially relevant example for which the time-delay represents a significant challenge for constraint enforcement. Future work will focus on extending the proposed method to address nonlinear dynamics, unknown time-varying delays, and non-uniform sample-data systems, e.g. [30], using more advanced Lyapunov-Krasovskii functionals.

REFERENCES