A Distributed Reference Governor for High-Order LTI Swarm Systems*

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Abstract—This paper focuses on the constrained control of high-order linear-time invariant (LTI) swarm systems. In particular, the system is first locally pre-stabilized using a proportional-integral control law and the stability property of the controlled system is proved using swarm stability arguments. Then, we show that, under certain conditions, the pre-stabilized system systematically admits a candidate Lyapunov function and an associate upper-bound function which is related to the constraints. Then, we propose a set invariance-based distributed Reference Governor (RG) to modify the reference such that the trajectories of the agents do not violate the constraints at each time instant. Furthermore, we show that the performance of the proposed RG scheme can be improved using a phase-lead compensator. Numerical examples are provided to demonstrate the effectiveness of the proposed method.

I. INTRODUCTION

In recent years, the study of swarm systems has attracted the interest of several researchers around the world. Applications include flocking in biological systems [1–3], self-driven particles in physical systems [4], formation control [5], and robotic swarm control [6–8] in engineering.

The constrained control of swarm systems is a relatively young topic. In practice, those constraints may appear in different forms depending on the system considered (e.g. geometric constraints, obstacles, input saturations, etc.). The importance of enforcing constraints satisfaction at all times is to ensure that safety, reliability, and stability properties of the controlled system are maintained. However, the main complexity of such systems is that they are in general subject to global constraints (i.e. constraints that depend on all the variables of the network), which must be dealt with in a distributed fashion. One of the many possible ways to deal with these constraints is presented in [9], where the proposed solution is based on the Reference Governor (RG).

The RG philosophy is used in various applications [10–14], where the main idea is to pre-stabilize the system and then to manage the constraints using the RG by suitably changing the applied reference. In particular, the set invariance-based RG has become very popular in recent days due to its very fast computational capability and its simplicity [15, 16]. This paper uses the set invariance-based distributed RG introduced in [17], which is based on very recent works that are using passivity arguments combined with convex optimization theory to solve general optimization problems [18].

In this paper, we deal with the constrained control of consensus-based linear-time invariant (LTI) high-order swarm systems. This paper is an extension of [17], where the results are extended to LTI high-order swarm systems considering directed and weighted interaction topologies. It is worth to mention that very recent works deal with high-order models [19–21] and even nonlinear dynamics [22] but do not consider constraints, which is the main contribution of this paper. Furthermore, we improve the performance of the scheme proposed in [17] by adding a phase-lead compensator, where its use is explored mostly in resource allocation optimization problems [23, 24]. In this paper, we show that the phase-lead control approach presented in these papers is also relevant for the problem considered here.

This paper is organized as follows. First, the problem of the constrained LTI swarm system is described. Then, the system is locally pre-stabilized using a proportional-integral control law. Afterwards, we augment the scheme with the distributed RG using a phase-lead compensator, where the convergence of the state estimates to the optimal solution is proved using passivity arguments. Finally, numerical examples are provided to demonstrate the effectiveness of the proposed solution.

II. PRELIMINARIES

This section gives the definitions of swarm stability, asymptotic swarm stability, and Lyapunov stability introduced in [19].

Definition 1. Consider an LTI swarm system composed of \( n \) agents, where \( x_1, x_2, \ldots, x_n \in \mathbb{R}^d \) are the agent states. The system is swarm stable if for \( \forall \epsilon > 0, \exists \delta(\epsilon) > 0 \) s.t. \( \|x_i(t) - x_j(t)\| < \epsilon \ (t > 0) \) as \( \|x_i(0) - x_j(0)\| < \delta(\epsilon) \) (\( \forall i, j \in \{1, 2, \ldots, n\} \)).

Definition 2. If an LTI swarm system achieves full state consensus, i.e. for \( \forall \epsilon > 0, \exists T > 0 \) s.t. when \( t > T \),
\[ \| x_i(t) - x_j(t) \| < \epsilon \quad (\forall i, j \in \{1, 2, \ldots, n\}), \] then it is asymptotically swarm stable.

**Definition 3.** An LTI system is Lyapunov stable if 0 is a stable point of equilibrium of this system.

**Remark 1.** A swarm stable system is not necessarily Lyapunov stable. Consider the counterexample where we assume the trajectory of a swarm stable system to be expressed by the average of the agents \( \xi(t) \) with dynamics \( \dot{\xi}(t) = A_\xi(t) \). In the case \( A \) is unstable, the deviation from \( \xi(t) \) can be unbounded for a small perturbation on the initial conditions.

### III. Problem Statement

Consider a system composed of \( n \) agents \( \mathcal{V} = \{1, 2, \ldots, n\} \), where the state of agent \( i \) is denoted by \( [x_{i1}, x_{i2}, \ldots, x_{id}]^T \in \mathbb{R}^d \). The inter-agent communication is modeled by the time-invariant graph \( G \), where the arc weight of \( G \) between agents \( i \) and \( j \) is denoted by \( w_{ij} \geq 0 \) for \( i, j \in \mathcal{V} \). Then, we can denote \( G \) by its weighted adjacency matrix \( W \)

\[
W = \begin{bmatrix}
w_{11} & w_{12} & \cdots & w_{1n} \\
w_{21} & w_{22} & \cdots & w_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
w_{n1} & w_{n2} & \cdots & w_{nn}
\end{bmatrix},
\]

where \( w_{ij} \in \mathbb{R}^+ \), for \( i, j \in \mathcal{V} \). Accordingly, we define the non-weighted adjacency matrix \( W^n = \frac{W}{2} \) as

\[
W^n = \begin{bmatrix}
w_{11} & w_{12} & \cdots & w_{1n} \\
w_{21} & w_{22} & \cdots & w_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
w_{n1} & w_{n2} & \cdots & w_{nn}
\end{bmatrix},
\]

where \( w_{ii} = 1, w_{ij} = 1 \) if \( w_{ij} \neq 0 \), and \( w_{ii} = 0 \) otherwise \((i, j) \in \mathcal{V})\). Note that if \( G \) is undirected, then \( W = W^T \) and \( W^n = (W^n)^T \) are symmetric. In this paper, we use \( W \) to formulate the consensus between the agents. In this case, \( w_{ij} \) can be regarded as the strength of the information link. On the other hand, we use \( W^n \) to express the local knowledge of each agent. In particular, we assume that each agent knows its own state and its neighbors’ states.

The dynamics of the \( i \)-th agent is the \( d \)-th order consensus-based swarm system

\[
\dot{x}_i = Ax_i + F \sum_{j=1}^n w_{ij}(x_j - x_i) + Bu_i,
\]

\[
y_i = Cx_i,
\]

where \( i = \{1, 2, \ldots, n\}, A \in \mathbb{R}^{d \times d}, F \in \mathbb{R}^{d \times d}, B \in \mathbb{R}^{d \times q}, C \in \mathbb{R}^{p \times d}, y_i \in \mathbb{R}^p \) is the output of the system, and \( u_i \in \mathbb{R}^q \) the input of the system. Then, we assume the following.

**Assumption 1.** \((A, B)\) is controllable.

We furthermore assume that all the agents are subject to global constraints, which are polyhedral and are in the form

\[
c_x^T x_i(t) + d_x \geq 0, \forall t \in [0, \infty), \forall i \in \mathcal{V}
\]

\[
c_u^T u_i(t) + d_u \geq 0, \forall t \in [0, \infty), \forall i \in \mathcal{V}
\]

where \( c_x \in \mathbb{R}^{d \times n_x}, c_u \in \mathbb{R}^{q \times n_u}, d_x \in \mathbb{R}^{n_x}, d_u \in \mathbb{R}^{n_u}, n_x \in \mathbb{N}_0 \) is the number of constraints on \( x_i \), and \( n_u \in \mathbb{N}_0 \) the number of constraints on \( u_i \).

The objective of this paper is to synchronize \( y_i \) with the desired constant output reference \( r \in \mathbb{R}^p \) for \( \forall i \in \mathcal{V} \) while ensuring that constraints (5) and (6) are satisfied at all times.

Examples where \( r \) is used as an external input include leader reference tracking and external human command scenarios (e.g. [15], [17]). The first step is to pre-stabilize the system using a proportional-integral control law. Stability properties of the controlled system are proved using swarm stability arguments. Then, we augment the scheme with a distributed RG to enforce constraints satisfaction at each time instant.

### IV. Local Pre-Stabilization

This section proposes a local control law and studies the stability properties of the controlled system. To do so, we assume that \( x_h \in \mathbb{R}^d \) is the equilibrium that produces the output reference \( r \) and we define the state error \( \tilde{x}_i := x_h - x_i \in \mathbb{R}^d \). Then we propose the local control law

\[
u_i = Kg\tilde{x}_i + G\tilde{\xi}_i,
\]

where \( \tilde{\xi}_i := \int_{0}^{t} \tilde{x}_i dt \in \mathbb{R}^d \) is the integral term, \( K \in \mathbb{R}^{q \times d} \) the proportional gain matrix, and \( G \in \mathbb{R}^{n \times d} \) the integral gain matrix. Then, using (4) and (7) in (5), the aggregated system becomes

\[
\dot{\tilde{\eta}} = \left( I_n \otimes \begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix} - L \otimes \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \right) \tilde{\eta},
\]

where \( \tilde{\eta} := [\tilde{\eta}_1^T, \tilde{\eta}_2^T, \ldots, \tilde{\eta}_n^T]^T \in \mathbb{R}^{2nd} \) is the aggregated state with \( \tilde{\eta}_i := [\tilde{x}_i^T, \tilde{\xi}_i^T]^T \in \mathbb{R}^{2d} \), \( I \in \mathbb{R}^{\times \times} \) the identity matrix, \( \otimes : \mathbb{R}^{a \times b} \times \mathbb{R}^{c \times d} \rightarrow \mathbb{R}^{ac \times bd} \) the Kronecker product, and \( L = L(G) \) the Laplacian matrix associated to \( G \) [25]. At this point, we manipulate \( L \) using its associated Jordan canonical form \( J(\lambda_i) \)

\[
J(\lambda_i) = \begin{bmatrix} 0 & * & 0 & \cdots & 0 \\
0 & \lambda_2 & * & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix},
\]

where \( \lambda_1 = 0, \lambda_2, \ldots, \lambda_n \in \mathbb{C} \) are the eigenvalues of \( L \) and * can be replaced either by 1 or 0. Then, assuming that \( J = TLT^{-1} \) and making the change of coordinates \( \tilde{\eta}^J = (T \otimes I_d)\tilde{\eta} \), we can rewrite (8) as

\[
\dot{\tilde{\eta}}^J = \left( I_n \otimes \begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix} - J \otimes \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \right) \tilde{\eta}^J.
\]

The following lemma proves that System (10) is asymptotically stable and that \( \lim_{t \rightarrow \infty} y_i(t) = r \) \( (\forall i \in \mathcal{V}) \) for \( K \) and \( G \) properly chosen.

**Lemma 1.** Consider System (10) controlled by (7). Then, the system is asymptotically stable and \( \lim_{t \rightarrow \infty} y_i(t) = r \)
(∀i ∈ V) for K and G chosen such that

\[
\begin{bmatrix}
A + BK & BG \\
I_d & 0
\end{bmatrix}
\quad \text{and}
\quad
\begin{bmatrix}
A + BK - \lambda_i F & BG \\
I_d & 0
\end{bmatrix}
\quad (\lambda_i \neq 0)
\] are Hurwitz.

**Proof.** First, note that \((I_n \otimes \begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix}) - J \otimes \begin{bmatrix} F & 0 \end{bmatrix}\) can be rewritten in the form

\[
\begin{bmatrix}
A + BK & BG \\
I_d & 0
\end{bmatrix} \times \begin{bmatrix} 0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
A + BK - \lambda_i F & BG \\
I_d & 0
\end{bmatrix} \times \begin{bmatrix} 0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\]

where \(\times\) is either \([-F \\ 0] \) or \([0 \\ 0]\) depending on \(J\). Because of Assumption \(V\), it is possible to choose \(K\) and \(G\) such that

\[
\begin{bmatrix}
A + BK & BG \\
I_d & 0
\end{bmatrix} \quad \text{and}
\quad
\begin{bmatrix}
A + BK - \lambda_i F & BG \\
I_d & 0
\end{bmatrix}
\quad (\lambda_i \neq 0)
\] are Hurwitz. As a consequence, in view of (11), System (10) is swarm stable. Then, we can also choose \(K\) and \(G\) such that \(\begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix} \) is Hurwitz. As a result, using Lemma 3 of \(V\), it follows asymptotic swarm stability and, therefore, Lyapunov stability. At this point, it remains to prove that the system is asymptotically stable and that \(y_i \rightarrow r\) for \(t \rightarrow \infty\). To do so, first note that, since \(\begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix} \) and \(\begin{bmatrix} A + BK - \lambda_i F & BG \\ I_d & 0 \end{bmatrix} \) are Hurwitz, using Barbalat’s lemma, we can prove that \(\lim_{t \rightarrow \infty} x_i(t) = x_h\) \((\forall i \in V)\) since \(\lim_{t \rightarrow \infty} \tilde{y}(t) = 0\). Therefore, asymptotic stability follows and, as a result, \(\lim_{t \rightarrow \infty} y_i = r\) for \(\forall i \in V\), which concludes the proof.

**Corollary 1.** If \(K\) and \(G\) are chosen such that

\[
\begin{bmatrix}
A + BK & BG \\
I_d & 0
\end{bmatrix}
\quad \text{and}
\quad
\begin{bmatrix}
A + BK - \lambda_i F & BG \\
I_d & 0
\end{bmatrix}
\quad (\lambda_i \neq 0)
\] admits the quadratic Lyapunov function

\[
V(\tilde{y}) = \tilde{y}^T P \tilde{y},
\]

where \(P = P^T > 0\) can be freely chosen such that

\[
(I_n \otimes \begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix}) - J \otimes \begin{bmatrix} F & 0 \end{bmatrix} \times \begin{bmatrix} 0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix} + P (I_n \otimes \begin{bmatrix} A + BK & BG \\ I_d & 0 \end{bmatrix} - J \otimes \begin{bmatrix} F & 0 \end{bmatrix}) Q = 0,
\]

with \(Q = Q^T > 0\). Moreover, since the system is asymptotically stable, \(V(\tilde{y}) = -\tilde{y}^T Q \tilde{y} < 0\) is negative definite.

In the next section, the presented control scheme will be augmented with the passivity-based distributed RG introduced in [17].

**V. DISTRIBUTED REFERENCE GOVERNOR**

The proposed RG scheme uses a set invariance-based philosophy as proposed in \([15, 26]\). In this section, we formulate first the global optimization problem and then we derive the local optimization problem of the RG. Afterwards, we propose a passivity-based distributed solver inspired by [17] and then we study the convergence properties of the proposed scheme.

**A. Global Optimization Problem**

To formulate the global optimization problem, the first step is to introduce the modified reference which is used to ensure constraints satisfaction at all times. Then, we will manipulate the constraints (5–6) to upper-bound the Lyapunov level sets in order to formulate the RG problem as an optimization problem, subject to an inequality constraint expressed in terms of the level sets. To solve the inter-agent communication problem, we introduce a new optimization variable to estimate the system states that are locally unknown through an equality constraint.

Let us define \(m \in \mathbb{R}^d\) as the modified reference of \(x_h\),

\[
\tilde{x}_m := m - x_i \in \mathbb{R}^d\text{ the shifted state to } m, \quad \tilde{\eta}_m := \left(\tilde{\eta}_m^r T, \tilde{\eta}_m^l T\right)^T \in \mathbb{R}^{2nd}\text{ the shifted aggregated state to } m, \quad \tilde{\eta}_m := \left(\tilde{\eta}_m^r T, \tilde{\eta}_m^l T\right)^T \in \mathbb{R}^{2d}.
\]

Then, we aggregate (5) and (6) considering the control feedback (7) as

\[
c_m^T \tilde{\eta}_m + d_m \geq 0, \forall i \in V,
\]

where \(c_m \in \mathbb{R}^{2d \times (n_x + n_u)}\), and \(d_m \in \mathbb{R}^{n_x + n_u}\). Note that all the previous stability results hold true for \(\tilde{\eta}_m^r\) due to the linear properties of the controlled system. Therefore, we can still use (12) using the shifted state \(\tilde{\eta}_m\). Then, as shown in Proposition 1 of \([15]\), it is possible to bound the quadratic Lyapunov function of (12) as

\[
V(\tilde{\eta}_m) \leq \frac{(c_m^T \tilde{\eta}_m + d_m)^2}{c_l^T P^{-1} c_l} = \Gamma_l(\tilde{\eta}_m),
\]

for \(l \in \{1, 2, \ldots, n_x + n_u\}\), where \(c_l := (I_n \otimes I_{2d}) c_m \in \mathbb{R}^{2nd}\), \(1_n \in \mathbb{R}^n\) the unit column vector, \(c_m \in \mathbb{R}^{2d}\) is the l-th column of \(c_m\), and \(d_m \in \mathbb{R}\) the l-th element of \(d_m\).

At this point, we can formulate the global optimization problem as

\[
\min_m \|x_h - m\|^2
\]

subject to:

\[
g_l(\tilde{\eta}_m) \leq 0,
\]

where the l-th row of \(g_l(\tilde{\eta}_m)\) is \(g_l(\tilde{\eta}_m) := V(\tilde{\eta}_m) - \Gamma_l(\tilde{\eta}_m)\). Remark that (16)–(17) becomes a convex problem if \(g_l(\tilde{\eta}_m)\) is convex. The following lemma gives a necessary and sufficient condition for \(g_l(\tilde{\eta}_m)\) to be convex.

**Lemma 2.** Consider \(g_l(\tilde{\eta}_m)\) in \((17)\). Then, \(g_l(\tilde{\eta}_m)\) is convex if and only if \(P\) satisfies

\[
P - \frac{c_l c_l^T}{c_l^T P^{-1} c_l} \geq 0, \forall l \in \{1, 2, \ldots, n_x + n_u\}.
\]

**Proof.** Developing \(\Gamma_l(\tilde{\eta}_m)\) in \((15)\), we obtain

\[
\Gamma_l(\tilde{\eta}_m) = \frac{(\tilde{\eta}_m^r)^T (c_l c_l^T) \tilde{\eta}_m^l + 2 d_m c_l^T \tilde{\eta}_m^l + d_m^2}{c_l^T P^{-1} c_l}.
\]

Therefore, using (12) and (19), we can rewrite \(g_l(\tilde{\eta}_m)\) as

\[
\begin{align*}
g_l(\tilde{\eta}_m) &= (\tilde{\eta}_m)^T \left( P - \frac{c_l c_l^T}{c_l^T P^{-1} c_l} \right) \tilde{\eta}_m - \frac{2 d_m^2}{c_l^T P^{-1} c_l} + d_m^2. 
\end{align*}
\]
At this point, it is important to note that (16)-(17) neglects the inter-agent communication since \(g(\tilde{y}_i)\) is computed with the information of all the agents of the network. To solve the inter-agent communication problem, we need to introduce the new optimization variable \(z := [m^T, \tilde{y}_i^T]^T \in \mathbb{R}^{d+2nd}\), where \(z_i \in \mathbb{R}^{2nd}\) is the estimation of \(\tilde{y}_i^m\). Accordingly, we introduce the equality constraint \(h_i(z, \tilde{y}_i^m)\), where the \((i,j)\)-th element of \(h_i(z, \tilde{y}_i^m)\) is \(w_{ij}^T \cdot (z_i - \tilde{y}_j^m)(t)\). Therefore, we can rewrite (16)-(17) as

\[
\min_z f(z) \quad (21) \\
\text{subject to:} \quad g(z) \leq 0 \quad (22) \\
h_i(z, \tilde{y}_i^m) = 0, \quad (23)
\]

where \(f(z) := \sum_{i=1}^n \|x_h - m\|^2\). In the next subsection, the local problem is derived taking into account the local knowledge of the agents.

**B. Local Optimization Problem**

To derive the local problem, we break down the global problem (21)-(23) taking into account the point of view of the agent \(i\). Accordingly, the local problem is

\[
\min_{z_i} f_i(z_i) \quad (24) \\
\text{subject to:} \quad g(z_i) \leq 0 \quad (25) \\
h_i(z_i, \tilde{y}_i^m) = 0, \quad (26)
\]

where \(f_i(z_i) := \|x_h - m\|^2\) and \(h_i(z_i)\) is the \(i\)-th row of \(h(z)\). In the following, we denote the optimal solution of (21)-(23) as \(z^* \in \mathbb{R}^{d+2nd}\).

**C. Distributed Solver and Convergence Properties**

In this section, a local optimization-based scheme to solve (21)-(23) is proposed. The scheme is inspired by (17) and is improved using a phase-lead compensator. The proof of the convergence of the estimated states to the optimal solution uses passivity arguments assuming a static problem, i.e. \(\dot{x}_i = 0\) \((i \in V)\). Then, we assume the following.

**Assumption 2.** We assume that the system graph \(G\) is such that the PI consensus estimator [27] system is stable.

The first step is to define \(\chi := [\chi_m^T, \chi_{\tilde{y}}^T]^T \in \mathbb{R}^{d+2nd}\) as the estimation of \(z^*\). Then, we propose the following scheme:

\[
\chi_m = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \alpha \end{array} \right] \frac{0 \quad 0 \quad 0 \quad 0 \quad -\alpha}{0 \quad L \quad -\tilde{L} \quad \chi_{\tilde{y}} \quad -\alpha} \left[ \begin{array}{c} \phi_i(\chi) \\ \Psi_i(\chi) \end{array} \right] \quad (27)
\]

where \(\tilde{L} := L \otimes I_{2d}, \chi_{\tilde{y}} \in \mathbb{R}^{2nd}\) is the consensus integral variable of \(\chi_{\tilde{y}}\), \(\alpha \in \mathbb{R}_+^+\) the gain of the system, \(\phi_i(\chi) := \nabla f_i(\chi)\) the gradient of the local objective function, \(\mu_i := \nabla h_i(\chi)\) the estimate of the optimal Lagrange multiplier \(\mu^*\) associated to the inequality constraints, \(\hat{\nu} := \nabla g(\chi)\) the gradient of the inequality constraints, and \(\Omega(\chi) := \nabla g(\chi)\) the gradient of the equality constraints. The Lagrange multipliers \(\hat{\mu}\) and \(\hat{\nu}\) are updated as

\[
\hat{\mu} = \theta(\mu, g) \quad (28) \\
\hat{\nu} = h_i(\chi), \quad (29)
\]

where the \(l\)-th element of \(\theta\) is

\[
\theta_l(\mu_l, g_l) := \begin{cases} 0, & \text{if } \mu_l = 0 \text{ and } g_l < 0 \\ g_l, & \text{otherwise}. \end{cases} \quad (30)
\]

Note that, since the inequality and equality constraints are convex and linear, respectively, the optimal Lagrange multipliers \(\mu^*\) and \(\nu^*\) must satisfy the Karush-Kuhn-Tucker (KKT) conditions [28].

\[
\nabla f_i(z^*) + \nabla g^T(z^*)\mu^* + \nabla h_i^T(z^*)\nu^* = 0 \quad (31) \\
\mu^*_l \geq 0, \quad g_l(z^*) \leq 0, \quad \forall i \in V \quad (32) \\
g_l(z^*)\mu^*_l = 0, \quad \forall l = 1, 2, \ldots, n_a + n_u, \quad \forall i \in V. \quad (33)
\]

Accordingly, the initial condition \(\hat{\mu}(0)\) must satisfy \(\hat{\mu}(0) \geq 0\) in the update algorithm (28) to enforce \(\hat{\mu}(t) \geq 0\) at all times \(t\).

In this paper, we augment the proposed scheme with the phase-lead compensator \(M(s)\) (see Fig. 1), which is in the form

\[
M(s) := \frac{\tau s + 1}{a_1 \tau + 1} I_{d+4nd} \quad (34)
\]

where \(\tau \in \mathbb{R}_+^+\) and \(a \in (0, 1)\) are tunable parameters to shape the loop. At this point, we define \(\bar{v} := v - v^*\) as the input of

![Fig. 1. Distributed Reference Governor augmented with the phase-lead compensator \(M(s)\).](image-url)
$M(s)$ is linear and time invariant, the phase-lead compensator \( \tilde{\mathbf{A}} \) is still passive from $\dot{\mathbf{v}}$ to $[\chi^T , \zeta_\eta]^T$. \hfill \square

**Remark 2.** The introduction of a zero in the closed loop system can accelerate the response of the system. In fact, $M(s)$ can be used to shape \( \tilde{\mathbf{A}} \). To do so, classical tools in linear control theory can be used, where nonlinearities and MIMO properties can be neglected for design purposes.

The following theorem proves that the proposed scheme \( \tilde{\mathbf{A}} \) augmented with $M(s)$ converges to the optimal solution $z^*$. \hfill \square

**Theorem 1.** Consider solver \( \tilde{\mathbf{A}} \) augmented with $M(s)$ and the global optimization problem \( \tilde{\mathbf{A}} \) \( -(34) \). For $\dot{x}_i = 0$ ($i \in \mathcal{V}$), the estimated state $\chi$ asymptotically converges to $z^*$ for any $\alpha > 0$, for any $\tau > 0$, and for any $a \in (0,1)$.

**Proof.** The stability properties and the convergence to the optimal solution of the scheme proposed in \( \tilde{\mathbf{A}} \) \( \) hold true. In fact, $z^*$ remains the point of equilibrium of \( \tilde{\mathbf{A}} \) even with the introduction of $M(s)$. Then, following from Lemma 5 passivity properties of the gradient descent and consensus block (see Fig. 1) hold true since all the other elements are proved to be passive (see Lemma 8 of \( \) ). Finally, since passivity properties hold true, and because of the convexity properties of \( \tilde{\mathbf{A}} \), using hybrid Lasalle’s principle, it is possible to prove \( \) that the system is asymptotically converging to $z^*$, which concludes the proof. \hfill \square

The next section gives some numerical examples to demonstrate the effectiveness of the proposed method.

**VI. NUMERICAL EXAMPLES**

This section presents the results of three cases. The first case (Case 1) will show that the system is stable using control law \( \tilde{\mathbf{A}} \) but violates constraints during the transients without the use of the RG. The second case (Case 2) augments the scheme with the RG and demonstrates that the RG enforces constraints satisfaction at each time instant. The third case (Case 3) shows that the response of the system can be accelerated using a lead compensator.

Case 1 considers a system composed of 5 agents, which communicate their states $\tilde{h}_i \in \mathbb{R}^4$ through the weighted adjacency matrix

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.49 \\ 0.75 & 0 & 0 & 0 & 0.45 \\ 0 & 0 & 0 & 0.75 & 0 \\ 0.66 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.29 & 0 \\ \end{bmatrix}. \quad (35)$$

Agent $i$ is governed by \( \tilde{\mathbf{A}} \) with $B = C = I_2$ and

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 2 \\ \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 0.5 \\ \end{bmatrix}. \quad (36)$$

Note that $A$ is not Hurwitz implying that the system is unstable for $\eta_i = 0$. Therefore, we stabilize the system using the control law \( \tilde{\mathbf{A}} \), where

$$K = \begin{bmatrix} 1 & 2 \\ 3 & 0.5 \\ \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \end{bmatrix}. \quad (37)$$

Note that $G$ is chosen equal to zero since the system has already an integrator. We consider the constant output reference \( r = [1,2]^T \) and the state constraints \( \zeta = [-1,1]^T \) and $d = 3.5$. Therefore, the desired state $x_h$ associated to $r$ is simply $x_h = r$. The initial conditions of the agents are $x_1(0) = [1,0]^T$, $x_2(0) = [0,0]^T$, $x_3(0) = [1,1]^T$, $x_4(0) = [1,0]^T$, $x_5(0) = [2,0]^T$, and $\zeta_i = [0,0]^T$ ($i \in \mathcal{V}$).

Fig. 2 shows that the controlled system is asymptotically stable but, without the use of the RG, the constraints are violated during the transients.

Case 2 uses the same parameters as Case 1 but uses the RG scheme \( \tilde{\mathbf{A}} \), where $\alpha = 1$. The initial conditions of the estimated states are $\chi_{\tilde{\eta}}(0) = \tilde{h}'''(0)$ and $\chi_m = [0,0]^T$ for all the agents and the initial conditions for the Lagrange multipliers are $\mu(0) = 0$ and $\nu(0) = [0,\ldots,0]^T \in \mathbb{R}^n$. Fig. 3 shows that the states converge to the desired state $x_h$ without violating the constraints. The RG scheme \( \tilde{\mathbf{A}} \) modifies the

**Fig. 2.** Case 1: State trajectories without the RG. The states are converging to the desired state $x_h$, but violate the constraints during the transients.

**Fig. 3.** Case 2: State trajectories with the RG. The states are converging to the desired state $x_h$, without violating the constraints.
IV. EXPERIMENTAL RESULTS

Fig. 4. Case 2 and Case 3: Reference evolution of agent 1 with the RG. The response of the system can be accelerated with the use of the lead compensator.

VII. CONCLUSIONS

This paper has presented a distributed Reference Governor for a class of constrained LTI swarm systems. The main idea is to stabilize the system using a proportional-integral control law, where stability is proved using swarm stability arguments. Then, under certain conditions, it is possible to find a candidate Lyapunov function and an associate upper-bound function which is related to the constraints. The pre-stabilized system is then augmented with a set invariance-based distributed RG scheme, where the performance is improved using a phase-lead compensator. Future works will aim at extending the results to a class of nonlinear systems, where other relevant passive elements will be added to the system (e.g. time delays).

REFERENCES


